

Bounds on the Exponential Domination Number

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Abstract

As a natural variant of domination in graphs, Dankelmann et al. [Domination with exponential decay, Discrete Math. 309 (2009) 5877-5883] introduce exponential domination, where vertices are considered to have some dominating power that decreases exponentially with the distance, and the dominated vertices have to accumulate a sufficient amount of this power emanating from the dominating vertices. More precisely, if S is a set of vertices of a graph G , then S is an exponential dominating set of G if $\sum_{v \in S} \left(\frac{1}{2}\right)^{\text{dist}_{(G,S)}(u,v)-1} \geq 1$ for every vertex u in $V(G) \setminus S$, where $\text{dist}_{(G,S)}(u,v)$ is the distance between $u \in V(G) \setminus S$ and $v \in S$ in the graph $G - (S \setminus \{v\})$. The exponential domination number $\gamma_e(G)$ of G is the minimum order of an exponential dominating set of G .

Dankelmann et al. show

$$\frac{1}{4}(d+2) \leq \gamma_e(G) \leq \frac{2}{5}(n+2)$$

for a connected graph G of order n and diameter d . We provide further bounds and in particular strengthen their upper bound. Specifically, for a connected graph G of order n , maximum degree Δ at least 3, radius r at least 1, we show

$$\gamma_e(G) \geq \left(\frac{n}{13(\Delta-1)^2} \right)^{\frac{\log_2(\Delta-1)+1}{\log_2^2(\Delta-1)+\log_2(\Delta-1)+1}},$$

$$\gamma_e(G) \leq 2^{2r-2}, \text{ and}$$

$$\gamma_e(G) \leq \frac{43}{108}(n+2).$$

Keywords: domination, exponential domination

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1 Introduction

We consider finite, simple, and undirected graphs, and use standard notation and terminology.

A set D of vertices of a graph G is *dominating* if every vertex not in D has a neighbor in D . The *domination number* $\gamma(G)$ of G , defined as the minimum cardinality of a dominating set, is one of the most well studied quantities in graph theory [16]. As a natural variant of this classical notion, Dankelmann et al. [7] introduce exponential domination, where vertices are considered to have some dominating power that decreases exponentially with the distance, and the dominated vertices have to accumulate a sufficient amount of this power emanating from the dominating vertices. As a motivation of their model they mention information dissemination within social networks, where the impact of information decreases every time it is passed on.

Before giving the precise definitions for exponential domination, we mention three closely related well studied notions. A set D of vertices of a graph G is *k -dominating* for some positive integer k , if every vertex not in D has at least k neighbors in D [4, 5, 8, 11, 12, 14, 15, 19]. A set D of vertices of a graph G is *distance- k -dominating* for some positive integer k , if for every vertex not in D , there is some vertex in D at distance at most k [1–3, 13, 17, 20]. Finally, in *broadcast domination* [6, 9, 10, 18], each vertex v is assigned an individual dominating power $f(v)$ and dominates all vertices at distance at least 1 and at most $f(v)$. Exponential domination shares features with these three notions; similarly as in k -domination, several vertices contribute to the domination of an individual vertex, similarly as in distance- k -domination, vertices dominate others over some distance, and similarly as in broadcast domination, different dominating vertices contribute differently to the domination of an individual vertex depending on the relevant distances.

We proceed to the precise definitions, and also recall some terminology.

Let G be a graph. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The order $n(G)$ of G is the number of vertices of G , and the size $m(G)$ of G is the number of edges of G . For two vertices u and v of G , let $\text{dist}_G(u, v)$ be the distance in G between u and v , which is the minimum number of edges of a path in G between u and v . If no such path exists, then let $\text{dist}_G(u, v) = \infty$. An endvertex is a vertex of degree at most 1. For a rooted tree T , and a vertex u of T , let T_u denote the subtree of T rooted in u that contains u as well as all descendants of u . A leaf of a rooted tree is a vertex with no children. For non-negative integers d_0, d_1, \dots, d_k , let $T(d_0, d_1, \dots, d_k)$ be the rooted tree of depth $k+1$ in which all vertices at distance i from the root have exactly d_i children for every i with $0 \leq i \leq k$. A rooted tree is binary if every vertex has at most two children, and a binary tree is full if every vertex other than the leaves has exactly two children. For a positive integer k , let $[k]$ be the set of positive integers at most k .

Let S be a set of vertices of G . For two vertices u and v of G with $u \in S$ or $v \in S$, let $\text{dist}_{(G,S)}(u, v)$ be the minimum number of edges of a path P in G between u and v such that S contains exactly one endvertex of P and no internal vertex of P . If no such path exists, then let $\text{dist}_{(G,S)}(u, v) = \infty$. Note that, if u and v are distinct vertices in S , then $\text{dist}_{(G,S)}(u, u) = 0$ and $\text{dist}_{(G,S)}(u, v) = \infty$.

For a vertex u of G , let

$$w_{(G,S)}(u) = \sum_{v \in S} \left(\frac{1}{2} \right)^{\text{dist}_{(G,S)}(u,v)-1},$$

where $\left(\frac{1}{2} \right)^\infty = 0$. Note that $w_{(G,S)}(u) = 2$ for $u \in S$.

If $w_{(G,S)}(u) \geq 1$ for every vertex u of G , then S is an *exponential dominating set* of G . The *exponential domination number* $\gamma_e(G)$ is the minimum order of an exponential dominating set of G , and an exponential dominating set of G of order $\gamma_e(G)$ is *minimum*. By definition, every

dominating set is also an exponential dominating set, which implies $\gamma_e(G) \leq \gamma(G)$ for every graph G .

The following summarizes the main results of Dankelmann et al. [7].

Theorem 1 (Dankelmann et al. [7]) *If G is a connected graph of diameter $\text{diam}(G)$, then*

$$\frac{1}{4}(\text{diam}(G) + 2) \leq \gamma_e(G) \leq \frac{2}{5}(n(G) + 2).$$

Dankelmann et al. [7] discuss the tightness of their bounds. They show that the lower bound is satisfied with equality for the path P_n of order n with $n \equiv 2 \pmod{4}$, and they construct a sequence of trees T for which $\frac{\gamma_e(T)}{n(T)+2}$ tends to $\frac{3}{8}$. Finally, they describe one specific tree T with $\frac{\gamma_e(T)}{n(T)+2} = \frac{144}{377} \approx 0.382$, and ask whether there are trees T with $\frac{144}{377} < \frac{\gamma_e(T)}{n(T)+2} \leq \frac{2}{5}$.

Note that the lower bound in Theorem 1 implies $\gamma_e(G) = \Omega(\log n(G))$ for graphs G of bounded maximum degree, because the diameter of such graphs is $\Omega(\log n(G))$. Our first result is a polynomial, and not just logarithmic, lower bound.

Theorem 2 *If G is a graph of maximum degree $\Delta(G)$ at least 3, then*

$$\gamma_e(G) \geq \left(\frac{n(G)}{13(\Delta(G) - 1)^2} \right)^{\frac{\log_2(\Delta(G)-1)+1}{\log_2^2(\Delta(G)-1)+\log_2(\Delta(G)-1)+1}}.$$

As our second result, we show that $\gamma_e(G)$ is not only lower bounded but in fact also upper bounded in terms of the diameter of G , or rather the radius of G .

Theorem 3 *If G is a connected graph of radius $\text{rad}(G)$ at least 1, then $\gamma_e(G) \leq 2^{2\text{rad}(G)-2}$.*

Surprisingly, the bound in Theorem 3 is tight as we show by constructing a suitable example.

As our third result, we improve the upper bound in Theorem 1 as follows.

Theorem 4 *If G is a connected graph, then $\gamma_e(G) \leq \frac{43}{108}(n(G) + 2)$.*

Note that $\frac{43}{108} \approx 0.398$.

All proofs and further discussion are postponed to the next section.

2 Proofs

Proof of Theorem 2: Let $\Delta = \Delta(G)$, and $\alpha = 1 - \frac{1}{\log_2(\Delta-1)+1}$. Let S be an exponential dominating set of G . Let H arise from G by removing all edges between vertices in S . Clearly, S is still an exponential dominating set of H .

Let $k = |S|$.

Let

$$\begin{aligned} A &= \{v \in V(G) \setminus S : \text{dist}_H(v, S) \leq \alpha \log_2(k)\} \text{ and} \\ B &= \{v \in V(G) \setminus S : \text{dist}_H(v, S) > \alpha \log_2(k)\}, \end{aligned}$$

where $\text{dist}_H(v, S) = \min\{\text{dist}_H(v, u) : u \in S\}$.

For $u \in S$, let

$$C(u) = \{v \in B : \text{dist}_H(v, u) \leq \log_2(k) + 2\}.$$

Since in a graph of maximum degree Δ , there are at most $\frac{\Delta}{\Delta-2} ((\Delta-1)^d - 1) \leq 3((\Delta-1)^d - 1)$ vertices at distance between 1 and d from any given vertex, we obtain

$$|A| \leq 3k((\Delta-1)^{\alpha \log_2(k)} - 1) = 3k^{\frac{\log_2^2(\Delta-1) + \log_2(\Delta-1) + 1}{\log_2(\Delta-1) + 1}} - 3k.$$

Let

$$\mathcal{R} = \{(u, v) : u \in S, v \in C(u)\}.$$

Since $(u, v) \in \mathcal{R}$ implies $\text{dist}_H(v, u) \leq \log_2(k) + 2$, we obtain that, for every u in S , there are at most $3((\Delta-1)^{\log_2(k)+2} - 1) \leq 3(\Delta-1)^2 k^{\log_2(\Delta-1)}$ vertices v with $(u, v) \in \mathcal{R}$, which implies

$$|\mathcal{R}| \leq 3k(\Delta-1)^2 k^{\log_2(\Delta-1)}.$$

If there is some v in B such that there are less than $\frac{1}{4}k^\alpha$ vertices u with $(u, v) \in \mathcal{R}$, then $\text{dist}_H(v, u') > \log_2(k) + 2$ for more than $k - \frac{1}{4}k^\alpha$ vertices u' in S . Since $v \in B$ implies $\text{dist}_H(v, S) > \alpha \log_2(k)$, we obtain

$$w_{(H,S)}(v) < \frac{1}{4}k^\alpha \left(\frac{1}{2}\right)^{\alpha \log_2(k)-1} + \left(k - \frac{1}{4}k^\alpha\right) \left(\frac{1}{2}\right)^{(\log_2(k)+2)-1} \leq \frac{1}{2} + \frac{1}{2} \frac{(k - \frac{1}{4}k^\alpha)}{k} < 1,$$

which is a contradiction. Hence, for every v in B , there are at least $\frac{1}{4}k^\alpha$ vertices u with $(u, v) \in \mathcal{R}$, which implies

$$|\mathcal{R}| \geq \frac{1}{4}k^\alpha |B|.$$

Combining the upper and the lower bound on $|\mathcal{R}|$, we obtain

$$|B| \leq 12(\Delta-1)^2 k^{\log_2(\Delta-1)+1-\alpha} = 12(\Delta-1)^2 k^{\frac{\log_2^2(\Delta-1) + \log_2(\Delta-1) + 1}{\log_2(\Delta-1) + 1}}.$$

Altogether, we obtain

$$\begin{aligned} n(G) &= |S| + |A| + |B| \\ &\leq k + 3k^{\frac{\log_2^2(\Delta-1) + \log_2(\Delta-1) + 1}{\log_2(\Delta-1) + 1}} - 3k + 12(\Delta-1)^2 k^{\frac{\log_2^2(\Delta-1) + \log_2(\Delta-1) + 1}{\log_2(\Delta-1) + 1}} \\ &\leq 13(\Delta-1)^2 k^{\frac{\log_2^2(\Delta-1) + \log_2(\Delta-1) + 1}{\log_2(\Delta-1) + 1}}, \end{aligned}$$

which implies the desired bound. \square

It is not difficult to improve the constant 13 in Theorem 2 by adding some technicalities. For the sake of simplicity, we decided not to do so.

Proof of Theorem 3: Since G has a rooted spanning tree T of depth at most $\text{rad}(G)$, and $\gamma_e(G) \leq \gamma_e(T)$, it suffices to show $\gamma_e(T) \leq 2^{2d-2}$ for a rooted tree T of depth d at least 1. The proof is by induction on the depth d of T .

If $d = 1$, then the root r of T forms an exponential dominating set of T , and hence, $\gamma_e(T) = 1 = 2^{2 \cdot 1 - 2}$. If $d = 2$, then four children of r form an exponential dominating set of T , and, if r

does not have four children, then the set of all its children forms an exponential dominating set of T . Hence, $\gamma_e(T) \leq 4 = 2^{2 \cdot 2 - 2}$, and we may assume that $d \geq 3$.

If S is a set of 2^{2d-2} parents of leaves of T , then, since the distance between any vertex in S and any other vertex of T is at most $2d - 1$, and $(\frac{1}{2})^{(2d-1)-1} |S| = 1$, the set S is an exponential dominating set of T . Hence, we may assume that the set S_0 of all parents of leaves of T has less than 2^{2d-2} elements, and is not an exponential dominating set of T .

Suppose that S_0 has at least $\frac{1}{8} \cdot 2^{2d-2}$ elements. Let u be any vertex of T . If u has depth at least $d - 2$, then some vertex in S_0 has distance at most 1 from u , which implies $w_{(T, S_0)}(u) \geq 1$. If u has depth at most $d - 3$, then the distance between u and any vertex in S_0 is at most $2d - 4$. Since $(\frac{1}{2})^{(2d-4)-1} |S_0| \geq 1$, we obtain $w_{(T, S_0)}(u) \geq 1$ also in this case, which implies the contradiction that S_0 is an exponential dominating set of T . Hence, S_0 has less than $\frac{1}{8} \cdot 2^{2d-2}$ elements.

For a vertex u of T , let $w(u) = w_{(T_u, S_0 \cap V(T_u))}(u)$. Let T_1 arise from T by removing every vertex u such that $w(u) \geq 1$ for every vertex v in $V(T_u)$. By the choice of S_0 , this construction implies that T_1 is a rooted tree of depth at most $d - 3$. Note that T_1 might have depth 0, that is, it may consist only of the root. By induction, T_1 has an exponential dominating set S_1 of order at most $\max\{1, 2^{2(d-3)-2}\}$. Now, $S_0 \cup S_1$ is an exponential dominating set of T of order at most $\frac{1}{8} \cdot 2^{2d-2} + \max\{1, 2^{2(d-3)-2}\} \leq 2^{2d-2}$, which completes the proof. \square

In order to show that the bound in Theorem 3 is tight, we need a simple observation concerning binary trees.

Lemma 5 *If T is a binary tree with root r , and S is a set of vertices of T , then $w_{(T, S)}(r) \leq 2$ with equality if and only if T contains a full binary subtree F with root r such that $V(F) \cap S$ is the set of leaves of F .*

Proof: The proof is by induction on the depth d of T . If $d = 0$ or $r \in S$, then the statement is trivial. Hence, we may assume that $d \geq 1$ and $r \notin S$. Let r_1, \dots, r_k for some $k \in [2]$ be the children of r . For $i \in [k]$, let T_i be the subtree of T rooted in r_i , and let $S_i = S \cap V(T_i)$. Since $w_{(T, S)}(r) = \frac{1}{2} \sum_{i=1}^k w_{(T_i, S_i)}(r_i)$, we obtain, by induction, that $w_{(T, S)}(r) \leq 2$ with equality if and only if $k = 2$, and $w_{(T_1, S_1)}(r_1) = w_{(T_2, S_2)}(r_2) = 2$. Now, $w_{(T_1, S_1)}(r_1) = w_{(T_2, S_2)}(r_2) = 2$ is equivalent with the existence of suitable full binary subtrees F_1 and F_2 as described in the statement. Since the existence of F_1 and F_2 is clearly equivalent with the existence of the subtree F as described in the statement, the proof is complete. \square

For some positive integer d , let the rooted tree T arise by attaching 2^{2d-2} disjoint copies of the full binary tree

$$T(\underbrace{2, \dots, 2}_d, 0)$$

of depth $d - 1$ to the root r of T . By Theorem 3, we have $\gamma_e(T) \leq 2^{2d-2}$. In fact, we are going to show that $\gamma_e(T) = 2^{2d-2}$. Therefore, let S be a minimum exponential dominating set of T that does not contain any leaf (notice that if S contains a leaf, then we could replace this leaf by its parent and still have an exponential dominating set). Assume that S contains a vertex u that is neither the root r nor a parent of a leaf. If S does not contain the parent v of some leaf v' of T_u , then, as v' must be dominated, we must have $w_{(T, S)}(v) \geq 2$. By Lemma 5, this implies in particular that the second child of v , which is a leaf, is in S , contradicting the fact that S does not contain any leaf of T . So S contains all the parents of the leaves of T_u , and we have $w_{(T_u, S \setminus \{u\})}(u) = 2$. Now, $S \setminus \{u\}$ is an exponential dominating set of T , a contradiction. Hence, $S \setminus \{r\}$ contains only parents of leaves. Suppose that $|S| < 2^{2d-2}$. This implies that $d \geq 2$, and that there is some child

x of the root r such that $S \cap V(T_x) = \emptyset$. Since S is an exponential dominating set, it follows in particular that S does not contain r . So we have $\text{dist}_T(u, v) = 2d - 1$ for every vertex u in S and every leaf v in $V(T_x)$, and we obtain $(\frac{1}{2})^{(2d-1)-1} |S| \geq 1$, that is, $|S| \geq 2^{2d-2}$, which is a contradiction. Altogether, it follows that $\gamma_e(T) = 2^{2d-2}$.

Our next goal is to prove Theorem 4.

Similarly as the proof of Theorem 1 in [7], the proof of Theorem 4 is based on an inductive argument that uses local reductions. Unfortunately, the non-local character of exponential domination makes it unlikely that a local approach can lead to a best-possible result. Even in order to achieve a very small improvement of the upper bound in Theorem 1, the approach makes it necessary to consider a large number of cases and specific configurations. Since our goal was rather to obtain a constant lower than $2/5$ for the upper bound in Theorem 1 than to obtain the best-possible result, we tried to limit the number of cases as much as possible for the sake of simplicity. There are several parts of our proof though, where further obvious improvements are possible at the cost of considering more cases.

In the next subsection we collect several auxiliary results, and in Subsection 2.2 we prove Theorem 4.

2.1 Auxiliary results

Lemma 6 *Let T be a tree.*

- (i) *If $\text{diam}(T) \leq 2$, then $\gamma_e(T) = 1$.*
- (ii) *If $\text{diam}(T) = 3$, then $\gamma_e(T) = 2$ and $n(T) \geq 4$.*
- (iii) *If $\text{diam}(T) = 4$, then let u be the central vertex of T . Let u have k neighbors that are endvertices and $\ell \geq 2$ neighbors that are not endvertices.*
 - (a) *If $\ell = 2$, then $\gamma_e(T) = 2$ and $n(T) \geq 5$.*
 - (b) *If $\ell = 3$, then $\gamma_e(T) \leq 3$ and $n(T) \geq 7$.*
 - (c) *If $\ell \geq 4$, then $\gamma_e(T) \leq 4$ and $n(T) \geq 9$.*

Proof: Since the proofs of (i) and (ii) are straightforward, we only give details for the proof of (iii). Since T has diameter 4, no single vertex forms an exponential dominating set of T , which implies $\gamma_e(T) > 1$. If $\ell \leq 3$, then the neighbors of u that are no endvertices form an exponential dominating set of T , which implies $\gamma_e(T) \leq \ell$. If $\ell \geq 4$, then four neighbors of u that are no endvertices form an exponential dominating set of T , which implies $\gamma_e(T) \leq 4$. Since $n(T) \geq 1 + k + 2\ell$, the lower bounds on the order of T follow. \square

For the rest of this subsection, let T be a tree of diameter at least 5. We root T in a vertex of maximum eccentricity, that is, the depth of T is at least 3.

Lemma 7 *Let u be a vertex of T , and let v_1, \dots, v_k be some children of u .*

If one of the following conditions (i) to (xvii) holds, then there is a tree T' with $n(T') < n(T)$ and $\gamma_e(T) \leq \gamma_e(T') + \frac{5}{13}(n(T) - n(T'))$.

- (i) *$k = 2$, and v_1 and v_2 are leaves.*
- (ii) *$k = 1$, and $T_{v_1} \cong T(1, 1, 0)$.*

- (iii) $k = 2$, v_1 is a leaf, and $T_{v_2} \cong T(1, 0)$.
- (iv) $k = 4$, and $T_{v_i} \cong T(1, 0)$ for $i \in [4]$.
- (v) $k = 2$, v_1 is a leaf, and $T_{v_2} \cong T(2, 1, 0)$.
- (vi) $k = 2$, v_1 is a leaf, and $T_{v_2} \cong T(3, 1, 0)$.
- (vii) $k = 6$, and $T_{v_i} \cong T(2, 1, 0)$ for $i \in [6]$.
- (viii) $k = 3$, and $T_{v_i} \cong T(3, 1, 0)$ for $i \in [3]$.
- (ix) $k = 4$, $T_{v_1} \cong T(3, 1, 0)$, and $T_{v_i} \cong T(2, 1, 0)$ for $i \in [4] \setminus [1]$.
- (x) $k = 4$, $T_{v_i} \cong T(3, 1, 0)$ for $i \in [2]$, and $T_{v_i} \cong T(2, 1, 0)$ for $i \in [4] \setminus [2]$.
- (xi) $k = 4$, $T_{v_1} \cong T(1, 0)$, and $T_{v_i} \cong T(2, 1, 0)$ for $i \in [4] \setminus [1]$.
- (xii) $k = 3$, $T_{v_1} \cong T(1, 0)$, and $T_{v_i} \cong T(3, 1, 0)$ for $i \in [3] \setminus [1]$.
- (xiii) $k = 4$, $T_{v_1} \cong T(1, 0)$, $T_{v_2} \cong T(3, 1, 0)$, and $T_{v_i} \cong T(2, 1, 0)$ for $i \in [4] \setminus [2]$.
- (xiv) $k = 4$, $T_{v_i} \cong T(1, 0)$ for $i \in [3]$, and $T_{v_4} \cong T(3, 1, 0)$.
- (xv) $k = 4$, $T_{v_i} \cong T(1, 0)$ for $i \in [3]$, and $T_{v_4} \cong T(2, 1, 0)$.
- (xvi) $k = 4$, $T_{v_i} \cong T(1, 0)$ for $i \in [2]$, and $T_{v_i} \cong T(2, 1, 0)$ for $i \in [4] \setminus [2]$.
- (xvii) $k = 4$, $T_{v_i} \cong T(1, 0)$ for $i \in [2]$, $T_{v_3} \cong T(3, 1, 0)$, and $T_{v_4} \cong T(2, 1, 0)$.

Proof: We consider different cases corresponding to the above conditions. In each case, we construct a suitable tree T' with $n(T') < n(T)$. Throughout the proof, let S' be a minimum exponential dominating set of T' .

If (i) occurs, then let $T' = T - v_2$. If $u \in S'$ or $u, v_1 \notin S'$, then S' is also an exponential dominating set of T . If $u \notin S'$ and $v_1 \in S'$, then $S = (S' \setminus \{v_1\}) \cup \{u\}$ is an exponential dominating set of T . We obtain $\gamma_e(T) \leq \gamma_e(T') + 0(n(T) - n(T'))$.

If (ii) occurs, then let $T' = T - V(T_{v_1})$. If w is the neighbor of v_1 in T_{v_1} , then $S' \cup \{w\}$ is an exponential dominating set of T . We obtain $\gamma_e(T) \leq \gamma_e(T') + \frac{1}{3}(n(T) - n(T'))$.

For the remaining cases, let $T' = T - \bigcup_{i=1}^k V(T_{v_i})$. We specify a vertex w and a set W with the following properties:

- If $u \notin S'$, then $S' \cup W$ is an exponential dominating set of T .
- If $u \in S'$, then $(S' \setminus \{u\}) \cup \{w\} \cup W$ is an exponential dominating set of T .
- $\gamma_e(T) \leq \gamma_e(T') + c(n - n')$ with $c \leq \frac{5}{13}$.

We leave it to the reader to verify the straightforward details.

- (iii): Let $w = v_1$ and $W = \{v_2\}$. We obtain $c = \frac{1}{3}$.
- (iv): Let $w = v_1$ and $W = \{v_2, v_3, v_4\}$. We obtain $c = \frac{3}{8}$.

- (v) or (vi): Let $w = v_1$ and let W be the set of children of v_2 . We respectively obtain $c = \frac{1}{3}$ and $c = \frac{3}{8}$.
- (vii) or (viii) or (ix) or (x): Let w be a child of v_1 and let W be the set of children of v_1, \dots, v_k except for w . We respectively obtain $c = \frac{11}{30}$, $c = \frac{8}{21}$, $c = \frac{8}{22}$, and $c = \frac{9}{24}$.
- (xi) or (xii) or (xiii): Let w be v_1 and let W be the set of children of v_2, \dots, v_k . We respectively obtain $c = \frac{6}{17}$, $c = \frac{6}{16}$, and $c = \frac{7}{19}$.
- (xiv) or (xv): Let $w = v_1$ and let W be the set containing v_2, v_3 as well as the children of v_4 . We respectively obtain $c = \frac{5}{13}$ and $c = \frac{4}{11}$. See Figure 1 for an illustration of case (xiv).
- (xvi) or (xvii): Let $w = v_1$ and let W be the set containing v_2 as well as the children of v_3 and v_4 . We respectively obtain $c = \frac{5}{14}$ and $c = \frac{6}{16}$.

Note that the factor $\frac{5}{13}$ comes from case (xiv). The other cases actually lead to smaller factors. \square

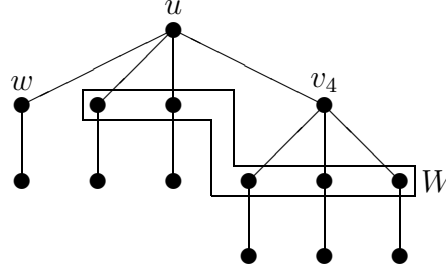


Figure 1: The configuration in case (xiv).

Lemma 8 *Let w be a vertex of T , and let X be a set of some children of w such that T_x has depth at most 2 for every x in X , and T_x has depth 2 for at least one x in X .*

If there is no tree T' with $n(T') < n(T)$ and $\gamma_e(T) \leq \gamma_e(T') + \frac{7}{18}(n(T) - n(T'))$, then there are non-negative integers k_1, k_2 , and k_3 such that

- $k_1 + k_2 + k_3 = |X|$.
- $T_x \cong T(1, 0)$ for k_1 vertices x in X .
- $T_x \cong T(2, 1, 0)$ for k_2 vertices x in X .
- $T_x \cong T(3, 1, 0)$ for k_3 vertices x in X .
- Furthermore, k_1, k_2 , and k_3 satisfy the following restrictions.
 - (a) $k_3 \leq 2$, and if $k_3 = 2$, then $(k_1, k_2, k_3) = (0, 0, 2)$.
 - (b) If $k_3 = 1$, then $k_2 \leq 2$.
 - (c) If $k_1 \geq 1$ and $k_3 = 1$, then $k_1 \leq 2$ and $k_2 \leq 1$.
 - (d) If $k_2 = 1$ and $k_3 = 1$, then $k_1 \leq 1$.
 - (e) If $k_3 = 0$, then $k_2 \leq 5$.
 - (f) If $k_1 \geq 1$ and $k_3 = 0$, then $k_2 \leq 2$.

(g) If $k_1 \geq 1$, $k_2 = 2$, and $k_3 = 0$, then $k_1 = 1$.

(h) If $k_1 \geq 1$, $k_2 = 1$, and $k_3 = 0$, then $k_1 \leq 2$.

Proof: Since $\frac{5}{13} < \frac{7}{18}$, we may assume, by Lemma 7, that T does not contain any of the substructures described in that lemma. By Lemma 7(i), $T_x \cong T(1, 0)$ for every x in X such that T_x has depth 1. Let k_1 be the number of x in X such that $T_x \cong T(1, 0)$. By Lemma 7(i) to (iv), $T_x \cong T(2, 1, 0)$ or $T_x \cong T(3, 1, 0)$ for every x in X such that T_x has depth 2. Let k_2 and k_3 be the numbers of x in X such that $T_x \cong T(2, 1, 0)$ and $T_x \cong T(3, 1, 0)$, respectively. Since T_x has depth 2 for at least one x in X , we have $k_2 + k_3 \geq 1$. By Lemma 7(iii), (v), and (vi), T_x has depth at least 1 for every x in X , which implies $k_1 + k_2 + k_3 = |X|$. By Lemma 7(viii), we have $k_3 \leq 2$.

Suppose now that $k_3 = 2$. By Lemma 7(x) and (xii), $k_1 = 0$ and $k_2 \leq 1$, which implies $(k_1, k_2, k_3) \in \{(0, 0, 2), (0, 1, 2)\}$. If $(k_1, k_2, k_3) = (0, 1, 2)$, then let T' arise from T by removing all descendants of w except for one child x of w . Let S' be a minimum exponential dominating set of T' . Clearly, we may assume that $x \notin S'$. Let Y be the set of the eight descendants of w at distance 2 from w . Let y be a vertex in Y with a neighbor of degree 4. See Figure 2 for an illustration.

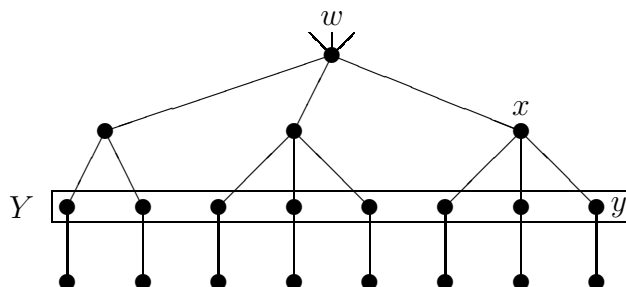


Figure 2: $(k_1, k_2, k_3) = (0, 1, 2)$.

If $w \notin S'$, then $w_{(T', S')}(x) \geq 1$ implies $w_{(T', S')}(w) \geq 2$, and hence, $S' \cup (Y \setminus \{y\})$ is an exponential dominating set of T . If $w \in S'$, then $(S' \setminus \{w\}) \cup Y$ is an exponential dominating set of T . In both cases, $\gamma_e(T) \leq \gamma_e(T') + \frac{7}{18}(n(T) - n(T'))$. Hence, $k_3 = 2$ implies $(k_1, k_2, k_3) = (0, 0, 2)$.

If $k_3 = 1$, then Lemma 7(ix) implies $k_2 \leq 2$. If $k_1 \geq 1$ and $k_3 = 1$, then Lemma 7(xiii) and (xiv) imply $k_2 \leq 1$ and $k_1 \leq 2$. If $k_2 = 1$ and $k_3 = 1$, then Lemma 7(xvii) implies $k_1 \leq 1$. If $k_3 = 0$, then Lemma 7(vii) implies $k_2 \leq 5$. If $k_1 \geq 1$ and $k_3 = 0$, then Lemma 7(xi) implies $k_2 \leq 2$. If $k_1 \geq 1$, $k_2 = 2$, and $k_3 = 0$, then Lemma 7(xvi) implies $k_1 = 1$. If $k_1 \geq 1$, $k_2 = 1$, and $k_3 = 0$, then Lemma 7(xv) implies $k_1 \leq 2$. \square

Note that in Lemma 7 and Lemma 8 we consider only some and not necessarily all children of u and w , respectively.

A vertex w of T has *type* (k_1, k_2, k_3) for non-negative integers k_1 , k_2 , and k_3 with $k_2 + k_3 \geq 1$, if

- k_1 , k_2 , and k_3 satisfy the restrictions stated in Lemma 8(a) to (h),
- w has exactly $k_1 + k_2 + k_3$ children,
- $T_x \cong T(1, 0)$ for k_1 children x of w ,
- $T_x \cong T(2, 1, 0)$ for k_2 children x of w , and
- $T_x \cong T(3, 1, 0)$ for k_3 children x of w .

Note that if w has some type, then T_w has depth 3.

Lemma 9 *Let the vertex w of T have type (k_1, k_2, k_3) .*

If $(k_1, k_2, k_3) \notin \{(0, 0, 2), (1, 0, 1), (2, 0, 1), (2, 1, 0)\}$, then there is a tree T' with $n(T') \leq n(T) - 6$ and $\gamma_e(T) \leq \gamma_e(T') + \frac{7}{18}(n(T) - n(T'))$.

Proof: By definition, k_1, k_2 , and k_3 satisfy the restrictions stated in Lemma 8(a) to (h). If $k_3 \geq 2$, then $(k_1, k_2, k_3) = (0, 0, 2)$. Hence, we may assume that $k_3 \leq 1$. We consider different cases. In what follows, T' will be a tree with $n(T') \leq n(T) - 6$, and S' will be a minimum exponential dominating set of T' . Let v be the parent of w .

Case 1 $k_3 = 1$.

In this case $k_2 \leq 2$.

If $k_1 = 0$, then let $T' = T - V(T_w)$, and let Y be the set of $2k_2 + 3$ descendants of w at distance 2 from w . The set $S' \cup Y$ is an exponential dominating set of T . Since $n(T') = n(T) - 5k_2 - 8$ and $\gamma_e(T) \leq \gamma_e(T') + 2k_2 + 3$, we obtain $\gamma_e(T) \leq \gamma_e(T') + \frac{7}{18}(n(T) - n(T'))$ as $2k_2 + 3 \leq \frac{7}{18}(5k_2 + 8)$. Hence, we may assume that $k_1 \geq 1$. Now, $k_1 \geq 1$ and $k_3 = 1$ imply $k_1 \leq 2$ and $k_2 \leq 1$.

If $k_2 = 0$, then $(k_1, k_2, k_3) \in \{(1, 0, 1), (2, 0, 1)\}$. Hence, we may assume that $k_2 = 1$, which implies $k_1 = 1$. Let x_1 and x_2 be the two children of w of degree at least 3, and let x_3 be the child of w of degree 2. Let $T' = T - (\{w\} \cup V(T_{x_1}) \cup V(T_{x_2})) + vx_3$. See Figure 3 for an illustration.

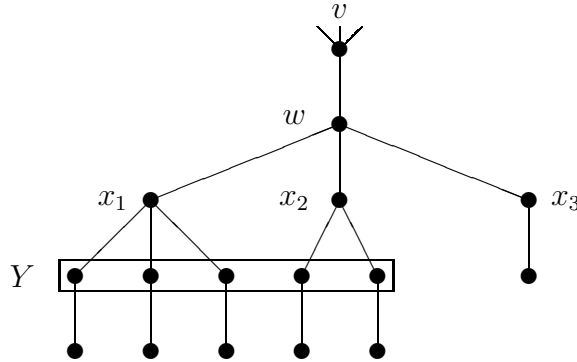


Figure 3: $(k_1, k_2, k_3) = (1, 1, 1)$.

Note that if S' contains neither x_3 nor the child of x_3 , then $w_{(T', S')}(v) \geq 4$. Let Y be the set of the five children of x_1 and x_2 . Since $S' \cup Y$ is an exponential dominating set of T , we obtain $\gamma_e(T) \leq \gamma_e(T') + \frac{5}{13}(n(T) - n(T'))$.

Case 2 $k_3 = 0$.

In this case $k_2 \leq 5$.

If $k_1 = 0$, then let $T' = T - V(T_w)$, and let Y be the set of the $2k_2$ descendants of w at distance 2 from w . The set $S' \cup Y$ is an exponential dominating set of T . Since $n(T') = n(T) - 5k_2 - 1$ and $\gamma_e(T) \leq \gamma_e(T') + 2k_2$, we obtain $\gamma_e(T) \leq \gamma_e(T') + \frac{5}{13}(n(T) - n(T'))$ as $2k_2 \leq \frac{5}{13}(5k_2 + 1)$ for $k_2 \leq 5$. Note that $n(T) - n(T') = 6$ only for $k_2 = 1$. Hence, we may assume that $k_1 \geq 1$, which implies $k_2 \leq 2$.

Case 2.1 $k_2 = 2$.

In this case $k_1 = 1$. Let x_1 and x_2 be the two children of w of degree 3, and let x_3 be the child of w of degree 2. Let $T' = T - (\{w\} \cup V(T_{x_1}) \cup V(T_{x_2})) + vx_3$. Let Y be the set of the four children of x_1 and x_2 . Since $S' \cup Y$ is an exponential dominating set of T , we obtain $\gamma_e(T) \leq \gamma_e(T') + \frac{4}{11}(n(T) - n(T'))$.

Case 2.2 $k_2 = 1$.

In this case $k_1 \leq 2$.

If $k_1 = 2$, then $(k_1, k_2, k_3) = (2, 1, 0)$. Hence, we may assume that $k_1 = 1$. Let $T' = T - V(T_w)$, and let X be the set containing the three parents of leaves in T_w . Since $S' \cup X$ is an exponential dominating set of T , we obtain $\gamma_e(T) \leq \gamma_e(T') + \frac{3}{8}(n(T) - n(T'))$. \square

A vertex of T is *good* if it has one of the types in $\{(0, 0, 2), (1, 0, 1), (2, 0, 1), (2, 1, 0)\}$.

Lemma 10 *If a vertex v of T has two children w_1 and w_2 such that w_1 has type $(2, 0, 1)$ and w_2 is good, then there is a tree T' with $n(T') < n(T)$ and $\gamma_e(T) \leq \gamma_e(T') + \frac{9}{23}(n(T) - n(T'))$.*

Proof: Let T' arise from $T - (V(T_{w_1}) \cup V(T_{w_2}))$ by adding the new vertex w , and adding the new edge vw . Let S' be a minimum exponential dominating set of T' . For each possible type of w_2 , we construct an exponential dominating set S of T from S' as follow: If $v \in S'$, then S is the union of $S' \setminus \{v\}$ and all parents of leaves of T_v , and if $v \notin S'$, then we add to S' all parents of leaves of T_v except for one child of w_1 to obtain S . We let the reader check that S is an exponential dominating set of T (using the fact that when $v \notin S'$, we must have $w_{(T', S')}(v) \geq 2$) and that we obtain the following results.

If w_2 has type $(2, 0, 1)$, then $n(T') = n(T) - 23$ and $\gamma_e(T) \leq \gamma_e(T') + 9$.

If w_2 has type $(1, 0, 1)$ or $(2, 1, 0)$, then $n(T') = n(T) - 21$ and $\gamma_e(T) \leq \gamma_e(T') + 8$.

If w_2 has type $(0, 0, 2)$, then $n(T') = n(T) - 26$ and $\gamma_e(T) \leq \gamma_e(T') + 10$. \square

Lemma 11 *If a vertex of T has three children that are good, then there is a tree T' with $n(T') < n(T)$ and $\gamma_e(T) \leq \gamma_e(T') + \frac{13}{33}(n(T) - n(T'))$.*

Proof: Suppose that v is a vertex of T that has three good children w_1 , w_2 , and w_3 . By Lemma 10, no child of v has type $(2, 0, 1)$. For each $(k_1, k_2, k_3) \in \{(1, 0, 1), (2, 1, 0), (0, 0, 2)\}$, let $n(k_1, k_2, k_3)$ vertices in $\{w_1, w_2, w_3\}$ have type (k_1, k_2, k_3) .

First, suppose that $n(1, 0, 1) \geq 2$ and say that w_1 and w_2 are of type $(1, 0, 1)$. Let T' arise from $T - (V(T_{w_1}) \cup V(T_{w_2}) \cup V(T_{w_3}))$ by adding the two new vertices w and x , and adding the two new edges vw and wx . From an exponential dominating set S' of T' , we construct an exponential dominating set of T . If w or x belongs to S' , then S is the union of $S' \setminus \{x, w\}$ and all the leaves of T_{w_1} and T_{w_2} . If w and x do not belong to S' , then v does not belong to S' also and S is obtained as the union of S' and all the leaves of T_{w_1} and T_{w_2} except for one child of w_1 . Notice that in this latter case we have $w_{(T', S')}(v) \geq 4$. In both cases we obtain the following results.

If $n(0, 0, 2) = 0$, then $n(T') = n(T) - 28$ and $\gamma_e(T) \leq \gamma_e(T') + 11$.

If $n(0, 0, 2) = 1$, then $n(T') = n(T) - 33$ and $\gamma_e(T) \leq \gamma_e(T') + 13$.

Hence, in these cases $\gamma_e(T) \leq \gamma_e(T') + \frac{13}{33}(n(T) - n(T'))$.

Next, suppose that either $n(1, 0, 1) = 1$ or $n(0, 0, 2) = 3$. Let T' arise from $T - (V(T_{w_1}) \cup V(T_{w_2}) \cup V(T_{w_3}))$ by adding the new vertex w , and adding the new edge vw . We derive as previously an exponential dominating set of T from an exponential dominating set of T' and obtain the following results.

If $n(1, 0, 1) = 1$, then considering the three possibilities for the other values, it follows that $\gamma_e(T) \leq \gamma_e(T') + \frac{15}{39}(n(T) - n(T'))$.

If $n(0, 0, 2) = 3$, then $n(T') = n(T) - 44$ and $\gamma_e(T) \leq \gamma_e(T') + 17$.

Hence, in these cases $\gamma_e(T) \leq \gamma_e(T') + \frac{17}{44}(n(T) - n(T'))$.

In what follows, we may assume that $n(1, 0, 1) = 0$ and $n(0, 0, 2) \leq 2$. Let $T' = T - (V(T_{w_1}) \cup V(T_{w_2}) \cup V(T_{w_3}))$. Once again we define an exponential dominating set of T from one of T' and obtain the following. We have $n(T') = n(T) - 10n(2, 1, 0) - 15n(0, 0, 2)$ and since $n(2, 1, 0) \geq 1$, it follows that $\gamma_e(T) \leq \gamma_e(T') + 4n(2, 1, 0) + 6n(0, 0, 2) - 1$. Considering the three possibilities for the value of $n(0, 0, 2)$ implies $\gamma_e(T) \leq \gamma_e(T') + \frac{15}{40}(n(T) - n(T'))$. \square

For the rest of this subsection, let v be a vertex of T such that

- T_v has depth 4,
- v has at most two children w such that T_w has depth 3,
- every child w of v such that T_w has depth 3 is good, and
- if v has two children that are good, then none of the two has type $(2, 0, 1)$.

Let W be the set of children w of v such that T_w has depth 3. Let $T^{(0)} = T - \bigcup_{w \in W} V(T_w)$, and let d_{red} be the depth of $T_v^{(0)}$. By construction, $d_{\text{red}} \leq 3$. Note that, since T has depth at least 5, the vertex v has a parent u in T .

Lemma 12 *If $d_{\text{red}} \leq 2$, then there is a tree T' with $n(T') < n(T)$ and $\gamma_e(T) \leq \gamma_e(T') + \frac{13}{33}(n(T) - n(T'))$.*

Proof: First, suppose that $d_{\text{red}} = 0$. Let $T' = T - V(T_v)$. If $|W| = 1$, then $\gamma_e(T) \leq \gamma_e(T') + \frac{5}{13}(n(T) - n(T'))$. If $|W| = 2$, then $\gamma_e(T) \leq \gamma_e(T') + \frac{12}{31}(n(T) - n(T'))$. Next, suppose that $d_{\text{red}} = 1$. Let T' arise from T by removing all descendants of v . For the two following cases we simply extend an exponential dominating set of T' by adding all the parents of the leaves of T_v . If $|W| = 1$, then $\gamma_e(T) \leq \gamma_e(T') + \frac{5}{13}(n(T) - n(T'))$. If $|W| = 2$, then $\gamma_e(T) \leq \gamma_e(T') + \frac{12}{31}(n(T) - n(T'))$.

Hence, we may assume that $d_{\text{red}} = 2$. Let v have n_1 children that are leaves, and n_2 children w such that T_w has depth 1. Since $d_{\text{red}} = 2$, we have $n_2 \geq 1$. By Lemma 7(i), we may assume that $T_w \cong T(1, 0)$ for every child w of v such that T_w has depth 1. We argue as previously for the following cases.

First, suppose that $n_2 = 1$. If v has a child w of type $(2, 0, 1)$, then w is the unique child of v such that T_w has depth 3. See Figure 4 for an illustration.

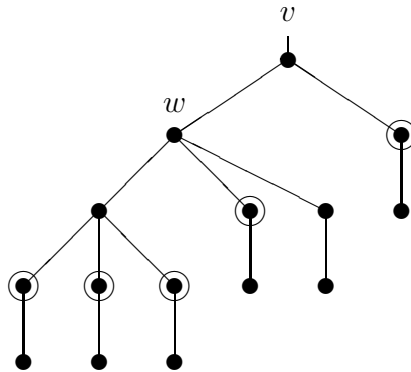


Figure 4: $W = \{w\}$, w has type $(2, 0, 1)$, $n_1 = 0$, and $n_2 = 1$.

In this case, let T' arise from T by removing all descendants of v except for w . Now, $n(T') = n(T) - 13 - n_1$ and $\gamma_e(T) \leq \gamma_e(T') + 5$, which implies $\gamma_e(T) \leq \gamma_e(T') + \frac{5}{13}(n(T) - n(T'))$. Hence, we may assume that v has no child of type $(2, 0, 1)$. Now, for $T' = T - V(T_v)$, it follows that $\gamma_e(T) \leq \gamma_e(T') + \frac{13}{33}(n(T) - n(T'))$.

Next, suppose that $n_2 \geq 3$. Let T' arise from T by removing all descendants of v . If v has exactly one child w such that T_w has depth 3, then $\gamma_e(T) \leq \gamma_e(T') + \frac{7}{18}(n(T) - n(T'))$. If v has two children w such that T_w has depth 3, then $\gamma_e(T) \leq \gamma_e(T') + \frac{14}{36}(n(T) - n(T'))$.

Finally, suppose that $n_2 = 2$. If v has a child w of type $(2, 0, 1)$, then w is the unique child of v such that T_w has depth 3. In this case, let T' arise from T by removing all descendants of v . Now, $n(T') = n(T) - 16 - n_1$ and $\gamma_e(T) \leq \gamma_e(T') + 6$, which implies $\gamma_e(T) \leq \gamma_e(T') + \frac{3}{8}(n(T) - n(T'))$. Hence, we may assume that v has no child of type $(2, 0, 1)$. Let T' arise from T by removing all descendants of v except for one child of v . It follows that $\gamma_e(T) \leq \gamma_e(T') + \frac{13}{33}(n(T) - n(T'))$. \square

2.2 Proof of Theorem 4

Since $\gamma_e(G) \leq \gamma_e(H)$ for every spanning subgraph H of G , it suffices to prove the statement in the case that G is a tree T . For a contradiction, suppose that T is a counterexample of minimum order. This choice of T implies that there is no tree T' with $n(T') < n(T)$ and $\gamma_e(T) \leq \gamma_e(T') + \alpha(n(T) - n(T'))$ for some $\alpha \leq \frac{43}{108}$. By Lemma 6, T has diameter at least 5. Root T in a vertex of maximum eccentricity. Let v be a vertex of T such that T_v has depth 4. Let W be the set of children w of v such that T_w has depth 3. By Lemma 8 and Lemma 9, every vertex in W is good. By Lemma 11, $|W| \leq 2$, and, by Lemma 10, if $|W| = 2$, then no vertex in W has type $(2, 0, 1)$. Let $T^{(0)} = T - \bigcup_{w \in W} V(T_w)$, and let d_{red} be the depth of $T_v^{(0)}$. By construction, $d_{\text{red}} \leq 3$, and Lemma 12 implies $d_{\text{red}} = 3$.

Now let X be the set of children x of v such that T_x has depth at most 2. By Lemma 8 applied to v and X , the vertex v has some type in the rooted tree $T^{(0)}$. Let $T^{(1)} = T - V(T_v)$. Let $T^{(2)}$ arise from T by removing all descendants of v . Finally, let $T^{(3)}$ arise from T by removing all descendants of v except for one child of v . As before we will extend a minimum exponential dominating set $S^{(i)}$ of some $T^{(i)}$ to obtain an exponential dominating set of T . We will use that $w_{(T^{(1)}, S^{(1)})}(u) \geq 1$ where u is the parent of v in T , that $w_{(T^{(2)}, S^{(2)})}(v) \geq 1$, and that $w_{(T^{(3)}, S^{(3)})}(v) \geq 2$ assuming that the child of v in $T^{(3)}$ does not belong to $S^{(3)}$. As before, for computations with $T^{(3)}$, we have to distinguish the cases $v \in S^{(3)}$ and $v \notin S^{(3)}$.

First, suppose that no vertex in W has type $(2, 0, 1)$. This implies that $n(T) - n(T^{(0)}) \leq 30$ and $\gamma_e(T) \leq \gamma_e(T^{(0)}) + \frac{2}{5}(n(T) - n(T^{(0)}))$ (by taking all the parents of the leaves of T_w for $w \in W$ to extend an exponential dominating set of $T^{(0)}$). If v is not good in $T^{(0)}$, then Lemma 9 implies that there is a tree T' with $n(T^{(0)}) - n(T') \geq 6$ and $\gamma_e(T^{(0)}) \leq \gamma_e(T') + \frac{7}{18}(n(T^{(0)}) - n(T'))$. If $0 < \alpha_1 < \alpha_2$, $1 \leq n_1^0 \leq n_1$, and $0 \leq n_2 \leq n_2^0$, then $\alpha_1 n_1 + \alpha_2 n_2 \leq (\frac{\alpha_1 n_1^0 + \alpha_2 n_2^0}{n_1^0 + n_2^0})(n_1 + n_2)$. Therefore,

$$\begin{aligned} \gamma_e(T) &\leq \gamma_e(T^{(0)}) + \frac{2}{5}(n(T) - n(T^{(0)})) \\ &\leq \gamma_e(T') + \frac{7}{18}(n(T^{(0)}) - n(T')) + \frac{2}{5}(n(T) - n(T^{(0)})) \\ &\leq \gamma_e(T') + \frac{\frac{7}{18} \cdot 6 + \frac{2}{5} \cdot 30}{6 + 30}(n(T) - n(T')) \\ &\leq \gamma_e(T') + \frac{43}{108}(n(T) - n(T')), \end{aligned}$$

which is a contradiction. Hence, v is good in $T^{(0)}$. If v is of type $(2, 0, 1)$ in $T^{(0)}$, then adding all parents of leaves in T_v except for one child of v to a minimum exponential dominating set of $T^{(2)}$ that does not contain v yields an exponential dominating set of T . This implies $\gamma_e(T) \leq \gamma_e(T^{(2)}) + \frac{16}{41}(n(T) - n(T^{(2)}))$, which is a contradiction (the worst case appears when $|W| = 2$ and each $w \in W$ is of type $(0, 0, 2)$). Hence, we may assume that v is not of type $(2, 0, 1)$ in $T^{(0)}$. It follows that $\gamma_e(T) \leq \gamma_e(T^{(3)}) + \frac{17}{43}(n(T) - n(T^{(3)}))$, which is a contradiction (the worst case appears when $|W| = 2$ and each $w \in W$ is of type $(0, 0, 2)$ and v is of type $(0, 0, 2)$ in $T^{(0)}$).

Hence, we may assume that one child w of v is of type $(2, 0, 1)$, which implies that w is the only element of W by Lemma 10. Let v have type (k_1, k_2, k_3) in $T^{(0)}$.

First, suppose $k_3 = 2$. This implies $(k_1, k_2, k_3) = (0, 0, 2)$ by Lemma 8(a), $n(T^{(1)}) = n(T) - 27$, and $\gamma_e(T) \leq \gamma_e(T^{(1)}) + 10$ by adding to a minimum exponential dominating set of $T^{(1)}$ all the parents of the leaves of T_v except for one child of w . So we have a contradiction and we assume that $k_3 \leq 1$.

Next, suppose $k_3 = 1$. In this case Lemma 8(b) implies $k_2 \leq 2$. If $k_2 = 1$, then adding all parents of leaves in T_v that are no children of v except for one child of w to a minimum exponential dominating set of $T^{(1)}$ yields an exponential dominating set of T . See Figure 5 for an illustration.

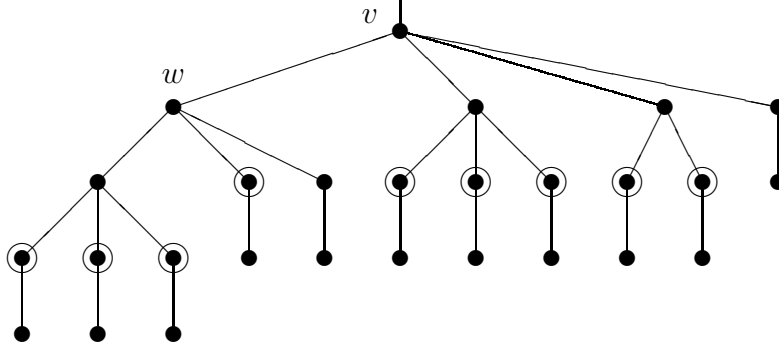


Figure 5: w has type $(2, 0, 1)$ in T , and v has type $(k_1, k_2, k_3) = (1, 1, 1)$ in $T^{(0)}$.

This implies $n(T^{(1)}) \leq n(T) - 25 - 2k_1$ and $\gamma_e(T) \leq \gamma_e(T^{(1)}) + 9$, which is a contradiction. Similarly, if $k_2 = 2$, then $n(T^{(1)}) \leq n(T) - 30 - 2k_1$ and $\gamma_e(T) \leq \gamma_e(T^{(1)}) + 11$, which is a contradiction. Hence, we may assume that $k_2 = 0$. By Lemma 7(xiv), we have $k_1 \leq 2$. If $k_1 = 1$, then $n(T^{(1)}) \leq n(T) - 22$ and $\gamma_e(T) \leq \gamma_e(T^{(1)}) + 8$, which is a contradiction. If $k_1 = 2$, then $n(T^{(1)}) \leq n(T) - 24$ and $\gamma_e(T) \leq \gamma_e(T^{(1)}) + 9$, which is a contradiction. Hence, we may assume that $k_1 = 0$. Now $n(T^{(3)}) = n(T) - 18$ and $\gamma_e(T) \leq \gamma_e(T^{(3)}) + 7$, which is also a contradiction. Hence, we may assume that $k_3 = 0$.

In this case Lemma 8(e) implies $1 \leq k_2 \leq 5$. If $k_1 = 0$, then we have $n(T^{(3)}) \leq n(T) - 5k_2 - 11$ and $\gamma_e(T) \leq \gamma_e(T^{(3)}) + 2k_2 + 4$ implying the contradiction $\gamma_e(T) \leq \gamma_e(T^{(3)}) + \frac{14}{36}(n(T) - n(T^{(3)}))$. Hence, we may assume that $k_1 \geq 1$. By Lemma 8(f), we have $k_2 \leq 2$. By Lemma 8(g), if $k_2 = 2$, then we have $k_1 = 1$. In this case we obtain $n(T^{(3)}) \leq n(T) - 23$ and $\gamma_e(T) \leq \gamma_e(T^{(3)}) + 9$, a contradiction. Finally, if $k_2 = 1$, then, by Lemma 8(h), we have $k_1 \leq 2$. If $k_1 = 1$, then $n(T^{(3)}) \leq n(T) - 18$ and $\gamma_e(T) \leq \gamma_e(T^{(3)}) + 7$, a contradiction. If $k_1 = 2$, then $n(T^{(2)}) \leq n(T) - 21$ and $\gamma_e(T) \leq \gamma_e(T^{(2)}) + 8$ implying the contradiction $\gamma_e(T) \leq \gamma_e(T^{(2)}) + \frac{8}{21}(n(T) - n(T^{(2)}))$, which completes the proof. \square

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